THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050 Mathematical Analysis (Spring 2018)
Tutorial on Mar 28

If you find any mistakes or typos, please email them to ypyang@math.cuhk.edu.hk

All through this notes, $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers.

Part I: Problems selected from the textbook

1. **(Ex 5.1.4)** Recall that the floor function is defined by $[x] := \max\{n \in \mathbb{Z} | n \leq x\}$. Determine the points of continuity of the following functions.

   (a) $f(x) = x[x]$,

   (b) $g(x) = \left[\dfrac{1}{x}\right] \quad (x \neq 0)$.

   Answer: (a) $f(x)$ is discontinuous at nonzero integers.

   (b) $g(x)$ is discontinuous at integers.

2. **(Ex 5.1.6)** Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$ be continuous at a cluster point $c \in A$. Show that for any $\varepsilon > 0$, there exists a neighborhood $V_\delta(c)$ of c such that whenever $x, y \in A \cap V_\delta(c)$ then it follows that $|f(x) - f(y)| < \varepsilon$.

   Proof: We can choose $\delta > 0$ such that if $x \in A \cap V_\delta(c)$, then $|f(x) - f(c)| < \dfrac{\varepsilon}{2}$. By triangle inequality, $x, y \in A \cap V_\delta(c) \Longrightarrow |f(x) - f(y)| \leq |f(x) - f(c)| + |f(y) - f(c)| < \varepsilon$.

3. **(Ex 5.1.9)** Let $A \subset B \subset \mathbb{R}$, let $f : B \to \mathbb{R}$ and let g be the restriction of f to A (that is, $g(x) = f(x)$ for $x \in A$).

   (a) If f is continuous at $c \in A$, show that g is continuous at c.

   (b) Show by example that if g is continuous at c, it need not follow that f is continuous at c.

   Proof: (a) Easy to prove. (b) Consider $f(x) = \operatorname{sgn}(x)$ on $B = [0, 1]$ and $g(x) = \operatorname{sgn}(x)$ on $A = (0, 1]$ and $c = 0$.

   Remark: Please relate this question with Problem 1 in Class Exercise 6.

4. **(Ex 5.1.11)** Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| < K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous at every point $c \in \mathbb{R}$.

   Proof: $\forall \varepsilon > 0$, we can take $\delta = \dfrac{\varepsilon}{K}$. Then $|x - c| < \delta \Longrightarrow |f(x) - f(c)| < K|x - c| < \varepsilon$.

   Remark: In this case, we say that f is **Lipschitz continuous** and K is a **Lipschitz constant**. We can see from above conclusion that Lipschitz continuity is a stronger property than continuity.

5. **(Ex 5.1.12, Ex 5.2.8)** Let $f(x) = 0$ for all $x \in \mathbb{Q}$ and suppose that f is continuous on $\mathbb{R}$. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.

   Proof: If $x \in \mathbb{I}$, then from the Density Theorem there exists a sequence $(x_n) \subset \mathbb{Q}$ that converges to x. Since f is continuous on $\mathbb{R}$, it follows that $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$.

   For Ex 5.2.8, we can apply above conclusion to $h(x) = f(x) - g(x)$.

6. **(Ex 5.1.13)** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x & \text{if } \ x \in \mathbb{Q} \\ x+3 & \text{if } \ x \in \mathbb{I} \end{cases}.$$

   Find the points of continuity of $f(x)$.

   Solution: $f(x)$ is continuous at $c = 3$ by noticing that

$$|f(x) - f(3)| = |f(x) - 6| \le \max(|2x - 6|, |x + 3 - 6|) = 2|x - 3|.$$

   If $c \neq 3$, we can choose two sequences $(x_n) \subset \mathbb{Q}$, $(y_n) \subset \mathbb{I}$ both convergent to c. Then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 2x_n = 2c, \quad \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} (y_n + 3) = c + 3.$$

   Since $c \neq 3$, the two limits are not equal and thus f is discontinuous at c.

7. Give an example for each of the following:

   (a) $f : \mathbb{R} \to \mathbb{R}$ continuous only at one point,

   (b) **(Ex 5.2.7)** $f : \mathbb{R} \to \mathbb{R}$ discontinuous everywhere but $|f|$ continuous everywhere,

   (c) $f : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{I}$ but discontinuous on $\mathbb{Q}$.

   Solution: (a) Question 2 in Part III.

   (b) $f(x) = \begin{cases} 1 & \text{if } \ x \in \mathbb{Q} \\ -1 & \text{if } \ x \in \mathbb{I} \end{cases}.$

   (c) The **Riemann function**.

8. **(Ex 5.2.12-13)** Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the **Cauchy equation** $f(x + y) = f(x) + f(y)$ for any $x, y \in \mathbb{R}$. Further suppose that there exists $x_0 \in \mathbb{R}$ at which f is continuous. Show that f is continuous everywhere and there exists some $c \in \mathbb{R}$ such that $f(x) = cx$, $\forall x \in \mathbb{R}$.

9. **(Optional)** Try to make use of the previous question and solve the following function equations. f is always assumed to be continuous on its domain.

   (a) **(Ex 5.2.14)** Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the relation $f(x + y) = f(x)f(y)$ for any $x, y \in \mathbb{R}$.
   Answer: If $f(c) = 0$ for some $c \in \mathbb{R}$, then $f(x) \equiv 0$. Otherwise, $f(x) = a^x$ where $a = f(1)$.

   (b) $f : (0, \infty) \to \mathbb{R}$ satisfies the relation $f(xy) = f(x)f(y)$, $\forall x, y > 0$.
   Answer: $f(x) = x^a$ or $f(x) \equiv 0$.

(c) $f:(0,\infty)\to\mathbb{R}$ satisfies the relation $f(xy)=f(x)+f(y)$, $\forall x,y>0$.
Answer: $f(x)=c\ln x$ where $c=f(e)$.

(d) $f:\mathbb{R}\to\mathbb{R}$ satisfies the relation $f(x+y)=f(x)e^y+f(y)e^x$, $\forall x,y\in\mathbb{R}$.
Answer: $f(x)=cxe^x$ where $c=\dfrac{f(1)}{e}$.

(e) $f:\mathbb{R}\to\mathbb{R}$ satisfies the relation $f(x+y)=f(x)+f(y)+xy$, $\forall x,y\in\mathbb{R}$.
Answer: $f(x)=\dfrac{x^2+cx}{2}$ where $c=2f(1)-1$.

(f) $f:\mathbb{R}\to\mathbb{R}$ satisfies the relation $f(xy)=yf(x)+xf(y)$, $\forall x,y\in\mathbb{R}$.
Answer: $f(x)=cx\ln x$ where $c=\dfrac{f(e)}{e}$.

(g) $f:\mathbb{R}\to\mathbb{R}$ satisfies the **Jensen equation** $f\left(\dfrac{x+y}{2}\right)=\dfrac{f(x)+f(y)}{2}$, $\forall x,y\in\mathbb{R}$.
Answer: $f(x)=cx+a$ where $a=f(0), c=f(1)-f(0)$.

Part II: Some comments.

1. There is no doubt that this chapter, especially Section 5.3, is the **MOST IMPORTANT** one of this course and we are studying the most important class of functions: continuous functions. Section 5.1 is the foundation of later sections and you should memorize the definition of continuity and its various equivalents. The examples should also be studied carefully. Section 5.2 is analogous to Section 4.2 and 3.2 but we will also study composite functions here, which enable us to establish the continuity of many functions. You can refer to Question 5 and 8 in Part III and obtain a deeper understanding to the continuity of composite functions.

2. **Definition of continuity**

   If $c\in A$ is an isolated point of A, then $f(x)$ is automatically continuous at c.

   If $c\in A$ is a cluster point of A, then $f(x)$ is continuous at c if and only if the following conditions hold:

   - f is defined at c (this condition is unnecessary when we only consider the limit of f at c),
   - the limit of f at c exists (in $\mathbb{R}$),
   - these two values are equal: $\lim\limits_{x\to c}f(x)=f(c)$.

   The continuity of $f(x)$ at some cluster point c requires that the limit of $f(x)$ at c exists and equals a specified value $f(c)$, and thus we just replace L by $f(c)$ in the definition of limit of a function:
   $$\forall\varepsilon>0,\ \exists\delta>0 \ \text{ such that } \ |x-c|<\delta\Longrightarrow|f(x)-f(c)|<\varepsilon.$$

3. **(Order preservation property, Ex 5.1.7 and Ex 5.2.10)** Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$ and $f(c)>0$, then $f(x)>0$ for x in some neighborhood of c.

Part III: Additional exercises.

1. **(Optional)(Difficult, for those who are interested)** Show that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ is continuous on $\mathbb{Q}$ but discontinuous on $\mathbb{I}$.

2. **(Question 2 in Part I on Mar 14 revisited)** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } \ x \in \mathbb{Q} \\ 0 & \text{if } \ x \in \mathbb{I} \end{cases}.$$

Determine the points of continuity of $f(x)$.

Proof: We have known that $\lim_{x\to 0} f(x) = 0 = f(0)$ and f does not have a limit at any nonzero numbers. So that $f(x)$ is only continuous at $x = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin \pi x & \text{if } \ x \in \mathbb{Q} \\ 0 & \text{if } \ x \in \mathbb{I} \end{cases}.$$

Find the points of continuity of $f(x)$.

Solution: If $c \notin \mathbb{Z}$, then we consider two sequences $(x_n) \subset \mathbb{Q}, (y_n) \subset \mathbb{I}$ both convergent to c. Then

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 0 = 0, \quad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin \pi x_n = \sin c\pi \neq 0.$$

Therefore, $\lim_{x\to c} f(x)$ does not exist and consequently f is not continuous at $c \notin \mathbb{Z}$.

If $c \in \mathbb{Z}$, then $\forall \varepsilon > 0$ we take $\delta = \frac{\varepsilon}{\pi}$. Whenever $|x - c| < \delta$, it follows that

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - 0| \leq |\sin \pi x| = |\sin(\pi x - \pi c + \pi c)| = |(-1)^c \sin(x - c)\pi| \\ &\leq |x - c|\pi < \varepsilon. \end{aligned}$$

Therefore, f is continuous at $c \in \mathbb{Z}$.

Remark: Compare this problem with the previous one.

4. Determine the points of continuity of **Riemann function (or Thomae's function)**

$$R(x) = \begin{cases} \dfrac{1}{q} & \text{if } \ x = \dfrac{p}{q} \in \mathbb{Q} \ \text{ where } \ p, q \in \mathbb{Z}, q > 0 \ \text{ and } \ \gcd(p, q) = 1, \\ 0 & \text{if } \ x \in \mathbb{I} \end{cases}.$$

Solution: We have shown that $\lim_{x\to c} f(x) = 0$, $\forall c \in \mathbb{R}$. Therefore, $R(x)$ is continuous at every irrational number and discontinuous at any rational number.

5. **(Question 4 in Part I on Mar 14 revisited, Ex 5.2.6)** Let $f, g : \mathbb{R} \to \mathbb{R}$ and $x_0, y_0 \in \mathbb{R}$. Suppose $\lim_{x\to x_0} g(x) = y_0$ and f is continuous at y_0, show that $\lim_{x\to x_0} f(g(x)) = f(y_0)$.

Proof: Given any $\varepsilon > 0$, because f is continuous at y_0, there exists $\delta_1 > 0$such that if $|y - y_0| < \delta_1$, then $|f(y) - f(y_0)| < \varepsilon$.

For this $\delta_1 > 0$, there exists $\delta > 0$ such that whenever $0 < |x - x_0| < \delta$ it follows that $|g(x) - y_0| < \delta_1$, which in turn implies that $|f(g(x)) - f(y_0)| < \varepsilon$.

Therefore, $\lim\limits_{x \to x_0} f(g(x)) = f(y_0)$.

Remark: Please compare this question with Question 7 below and also **Question 4 in Part I on Mar 14**. Notice that this question does not conclude that $f \circ g$ is continuous at x_0.

6. **(Question 1(b) in Part I on Mar 14 continued)** Let $f : A \to \mathbb{R}$ be bounded and c is a cluster point of A. Suppose $\lim\limits_{x \to c} f(x)$ does not exist in $\mathbb{R}$. Show that there exist two sequences $(x_n), (y_n) \subset A \setminus \{c\}$ and two real numbers L_1, L_2 such that

$$\lim_{n \to \infty} f(x_n) = L_1, \quad \lim_{n \to \infty} f(y_n) = L_2, \quad L_1 \neq L_2.$$

Proof: From **(Question 1(b) in Part I on Mar 14**, there exists $\varepsilon_0 > 0$ and two sequences $(a_n), (b_n)$ in $A \setminus \{c\}$, both convergent to c, such that $|f(a_n) - f(b_n)| \geq \varepsilon_0$, $\forall n$.

Since f is bounded, in particular $(f(a_n)), (f(b_n))$ are also bounded. Then by Bolzano-Weierstrass Theorem, there are $L_1, L_2 \in \mathbb{R}$ and subsequences $(f(a_{n_k}))$ of $(f(a_n))$ and $(f(b_{n_k}))$ of $(f(b_n))$ such that

$$\lim_{k \to \infty} f(a_{n_k}) = L_1, \quad \lim_{k \to \infty} f(b_{n_k}) = L_2.$$

From the Order-Preserving property, we have

$$|L_1 - L_2| = \lim_{k \to \infty} |f(a_{n_k}) - f(b_{n_k})| \geq \varepsilon_0 \Longrightarrow L_1 \neq L_2.$$

Denote $(a_{n_k}), (b_{n_k})$ as $(x_n), (y_n)$ respectively and we complete the proof.

7. **(On continuity of product)** From the theorem, if f, g are continuous at c, then so is fg. However, if they both are not continuous, there are various cases of continuity of fg. Discuss the following examples:

   (a) $f(x) = x$, $g(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

   (b) $f(x) = x$, $g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

   (c) $f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $g(x) = \begin{cases} 0, & x \geq 0 \\ 1, & x < 0 \end{cases}$.

   (d) $f(x) = g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

   Then we have the following results (check them yourself):

f at $c = 0$	g at $c = 0$	fg at $c = 0$	example
Continuous	Continuous	Continuous	Theorem
Continuous	Discontinuous	Continuous	(a)
Continuous	Discontinuous	Discontinuous	(b)
Discontinuous	Discontinuous	Continuous	(c)
Discontinuous	Discontinuous	Discontinuous	(d)

8. **(Question 5 in Part III on Mar 14 revisited)** For the following $g, f : \mathbb{R} \to \mathbb{R}$ and $x_0, y_0 = g(x_0) \in \mathbb{R}$, discuss the continuity of $g(x)$ at x_0, $f(y)$ at y_0 and $f(g(x))$ at x_0 respectively.

   (a) $g(x)$ is the Riemann function, $f(y)$ is the Dirichlet function and $x_0 = \pi, y_0 = 0$.
   $f(g(x)) \equiv 1$.

   (b) $g(x) = x^2$, $f(y) = \begin{cases} y & \text{if } \ y \leq 1 \\ y+1 & \text{if } \ y > 1 \end{cases}$ and $x_0 = 1, y_0 = 1$.
   $f(g(x)) = \begin{cases} x^2, & -1 \leq x \leq 1 \\ x^2 + 1, & x > 1 \ \text{ or } \ x < -1 \end{cases}$.

   (c) $g(x) = \operatorname{sgn}(x)$, $f(y) = y(1 - y^2)$ and $x_0 = 0, y_0 = 0$.
   $f(g(x)) \equiv 0$.

   (d) $g(x)$ is the Dirichlet function, $f(y) = y$ and $x_0 = 0, y_0 = 1$.
   $f(g(x))$ is the Dirichlet function.

   (e) Both of $g(x), f(y)$ are the Dirichlet function and $x_0 = 0, y_0 = 1$.
   $g(f(x)) \equiv 1$.

   (f) $g(x) = \begin{cases} \dfrac{1}{x} & \text{if } \ x \neq 0 \\ 0 & \text{if } \ x = 0 \end{cases}$, $f(y) = \operatorname{sgn}(y)$ and $x_0 = 0, y_0 = 0$. $g(f(x)) = \operatorname{sgn}(x)$.

Answer:

$g(x)$ at x_0	$f(y)$ at y_0	$f(g(x))$ at x_0	example
Continuous	Continuous	Continuous	Theorem
Continuous	Discontinuous	Continuous	(a)
Continuous	Discontinuous	Discontinuous	(b)
Discontinuous	Continuous	Continuous	(c)
Discontinuous	Continuous	Discontinuous	(d)
Discontinuous	Discontinuous	Continuous	(e)
Discontinuous	Discontinuous	Discontinuous	(f)

9. Suppose $f(x)$ is continuous on $[a, b]$ and there is a sequence $(x_n) \subset [a, b]$ such that $\lim_{n\to\infty} f(x_n) = A \in \mathbb{R}$. Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = A$. Does the conclusion still hold if $[a, b]$ is replaced by (a, b)?

   Solution: By **Bolzano-Weierstrass Theorem** (x_n) has a convergent subsequence (x_{n_k}). Suppose $\lim_{k\to\infty} x_{n_k} = x_0$, then $x_0 \in [a, b]$ (why?).

   Notice that $(f(x_{n_k}))$ is a subsequence of the convergent sequence $(f(x_n))$, and thus we have $f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = A$. Here the first identity is from the continuity of $f(x)$.

   If the closed interval is replaced by an open interval, the conclusion does not hold. Consider $f(x) = x$ defined on $(0, 1)$ and $x_n = \dfrac{1}{n}$, then $A = 0$ while there is no real number $x_0 \in (0, 1)$ with $f(x_0) = 0$.

10. Let $f : (a, b) \to \mathbb{R}$. Suppose $(f(x_n))$ is a Cauchy sequence for any Cauchy sequence (x_n) in (a, b), show that f is continuous on (a, b). Is the converse statement true or false?

    Solution: If otherwise $f(x)$ is not continuous on (a, b), then $f(x)$ is discontinuous at some $c \in (a, b)$. So $\exists \varepsilon > 0$ such that $\forall n \in \mathbb{N}$, there exists $x_n \in V_{\frac{1}{n}}(c)$ with $|f(x_n) - f(c)| \geq \varepsilon$.

    Let $(y_n) = (x_1, c, x_2, c, x_3, c, \cdots)$, then (y_n) is a Cauchy sequence since it converges to c. But $(f(y_n)) = (f(x_1), f(c), f(x_2), f(c), \cdots)$ is divergent and consequently not a Cauchy sequence.

    The converse is false. Consider $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ and $x_n = \dfrac{1}{n}$. Then f is continuous on $(0, 1)$ and (x_n) is a Cauchy sequence, but $(f(x_n)) = (n)$ is not a Cauchy sequence.